جـامـعـة بنهـــا

كلية الحاسبات والذكاء الإصطناعي

Probability and Statistics

Lecture 07

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Chapter 2: Random Variable

- Discrete Random Variables.
- Probability Mass Functions.
- Cumulative Distribution Functions.
- Discrete R.V. (Mean and Variance).
- Continuous Random Variables.
- Probability Density Functions.
- Continuous R.V. (Mean and Variance).
- Joint Probability Distributions.

Independence (1/3)

Independence:

If X and Y are two random variables, the X and Y are independent if any one of the following properties is true:

1) $f_{XY}(x, y) = f_X(x) f_Y(y)$ for all x and y

2) $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$

3) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$

Independence (2/3)

Example1:

$f_{XY}(x, y)$

y \ x	**1**	**1.5**	**2.5**	**3**
1	1/4	0	0	0
2	0	1/8	0	0
3	0	1/4	0	0
4	0	0	1/4	0
5	0	0	0	1/8

Find:

$f_X(x)$

$f_Y(y)$

Are X and Y independent?

Independence (3/3)

Example1 – Answer (1/3):

$f_{XY}(x, y)$

Find:

$f_X(x)$

y \ x	1	1.5	2.5	3
1	$1/4$	0	0	0
2	0	$1/8$	0	0
3	0	$1/4$	0	0
4	0	0	$1/4$	0
5	0	0	0	$1/8$
$f_X(x)$	$1/4$	$3/8$	$1/4$	$1/8$

Independence (3/3)

Example1 – Answer (1/3):

$f_{XY}(x, y)$

y \ x	1	1.5	2.5	3
1	1/4	0	0	0
2	0	1/8	0	0
3	0	1/4	0	0
4	0	0	1/4	0
5	0	0	0	1/8
$f_X(x)$	1/4	3/8	1/4	1/8

Find:

$f_X(x)$

Same as

x	1	1.5	2.5	3
$f_X(x)$	1/4	3/8	1/4	1/8

Independence (3/3)

Example1 – Answer (2/3):

$f_{XY}(x, y)$

y \ x	**1**	**1.5**	**2.5**	**3**	$f_Y(y)$
1	1/4	0	0	0	**1/4**
2	0	1/8	0	0	**1/8**
3	0	1/4	0	0	**1/4**
4	0	0	1/4	0	**1/4**
5	0	0	0	1/8	**1/8**

Find:

$f_Y(y)$

Independence (3/3)

Example1 – Answer (2/3):

$f_{XY}(x, y)$

y \ x	1	1.5	2.5	3	$f_Y(y)$
1	1/4	0	0	0	1/4
2	0	1/8	0	0	1/8
3	0	1/4	0	0	1/4
4	0	0	1/4	0	1/4
5	0	0	0	1/8	1/8

Find:

$f_Y(y)$

Same as

y	$f_Y(y)$
1	1/4
2	1/8
3	1/4
4	1/4
5	1/8

Independence (3/3)

Example1 – Answer (2/3):

$f_{XY}(x, y)$

Find:

$f_Y(y)$

y	$f_Y(y)$
1	1/4
2	1/8
3	1/4
4	1/4
5	1/8

Same as

y	1	2	3	4	5
$f_Y(y)$	1/4	1/8	1/4	1/4	1/8

Independence (3/3)

Example1 – Answer (3/3):

$f_{XY}(x, y)$

y \ x	**1**	**1.5**	**2.5**	**3**
1	1/4	0	0	0
2	0	1/8	0	0
3	0	1/4	0	0
4	0	0	1/4	0
5	0	0	0	1/8

Are X and Y independent?

Independence (3/3)

Example1 – Answer (3/3):

$f_{XY}(x,y)$

y \ x	1	1.5	2.5	3
1	1/4	0	0	0
2	0	1/8	0	0
3	0	1/4	0	0
4	0	0	1/4	0
5	0	0	0	1/8

Are X and Y independent?

If:

$f_{XY}(x,y) = f_X(x)f_Y(y)$ for all x and y

Independence (3/3)

Example1 – Answer (3/3):

$f_{XY}(x, y)$

y \ x	1	1.5	2.5	3
1	1/4	0	0	0
2	0	1/8	0	0
3	0	1/4	0	0
4	0	0	1/4	0
5	0	0	0	1/8

Same as

x	1	1.5	1.5	2.5	3
y	1	2	3	4	5
$f_{XY}(x, y)$	**1/4**	**1/8**	**1/4**	**1/4**	**1/8**

Independence (3/3)

Example1 – Answer (3/3):

x	1	1.5	1.5	2.5	3
y	1	2	3	4	5
$f_{XY}(x,y)$	1/4	1/8	1/4	1/4	1/8

x	1	1.5	2.5	3
$f_X(x)$	1/4	3/8	1/4	1/8

y	1	2	3	4	5
$f_Y(y)$	1/4	1/8	1/4	1/4	1/8

$$f_{XY}(1,1) = \frac{1}{4}$$

$$f_X(1)f_Y(1) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

Independence (3/3)

Example1 – Answer (3/3):

x	1	1.5	1.5	2.5	3
y	1	2	3	4	5
$f_{XY}(x,y)$	1/4	1/8	1/4	1/4	1/8

x	1	1.5	2.5	3
$f_X(x)$	1/4	3/8	1/4	1/8

y	1	2	3	4	5
$f_Y(y)$	1/4	1/8	1/4	1/4	1/8

$$f_{XY}(1,1) = \frac{1}{4} \quad \neq \quad f_X(1)f_Y(1) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

Then X and Y are not independent.

Linear Relationship (1/6)

Covariance (1/5):

The covariance between the random variables X and Y is the measure of *linear relationship* between them, denoted as $cov(X, Y)$ or σ_{XY}, where

$$cov(X, Y) = \sigma_{XY} = E\Big(\big(X - E(X)\big)\big(Y - E(Y)\big)\Big)$$

$$= E(XY) - E(X)E(Y)$$

Linear Relationship (1/6)

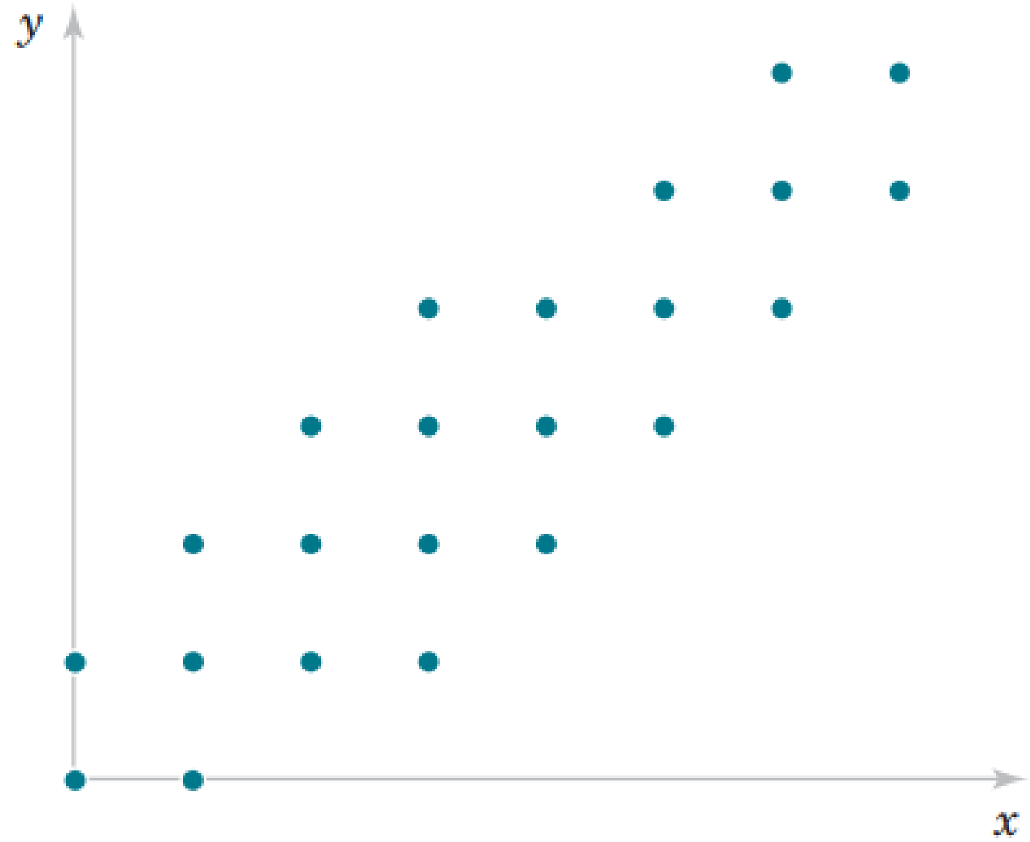
Covariance (2/5):

Covariance measures the total variation of two random variables from their expected values. Using covariance, we can only standard the direction of the relationship. However, it does not indicate the strength of the relationship.

Linear Relationship (1/6)

Covariance (3/5):

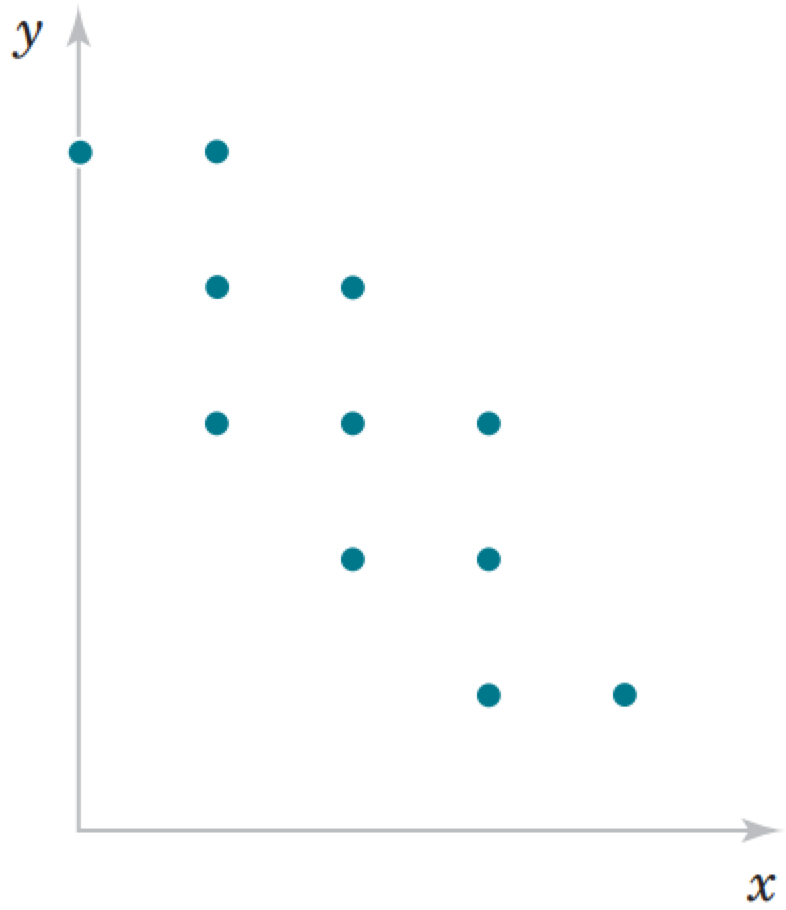
<u>Positive covariance</u>



Linear Relationship (1/6)

Covariance (4/5):

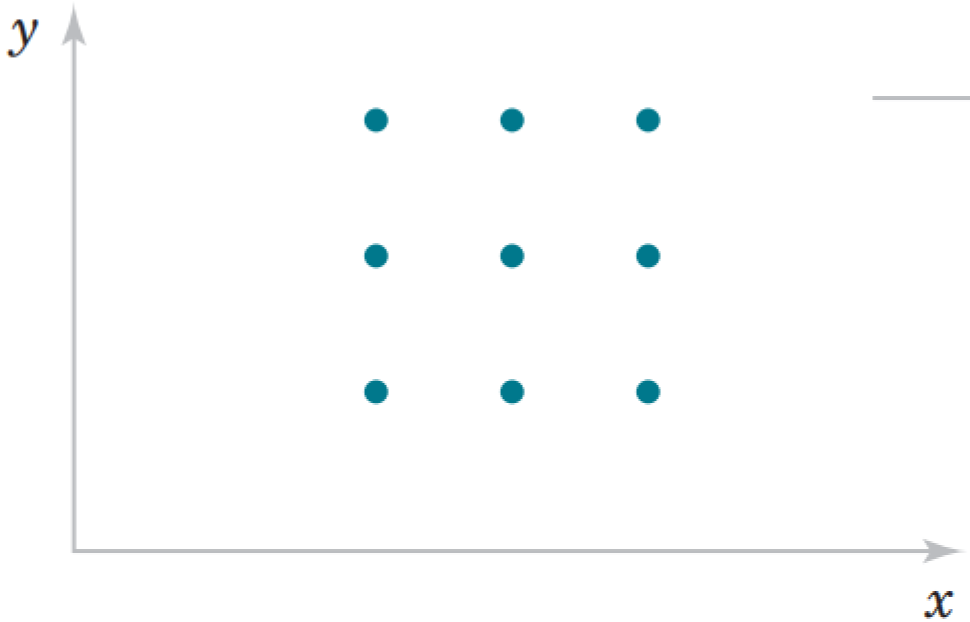
<u>Negative covariance</u>

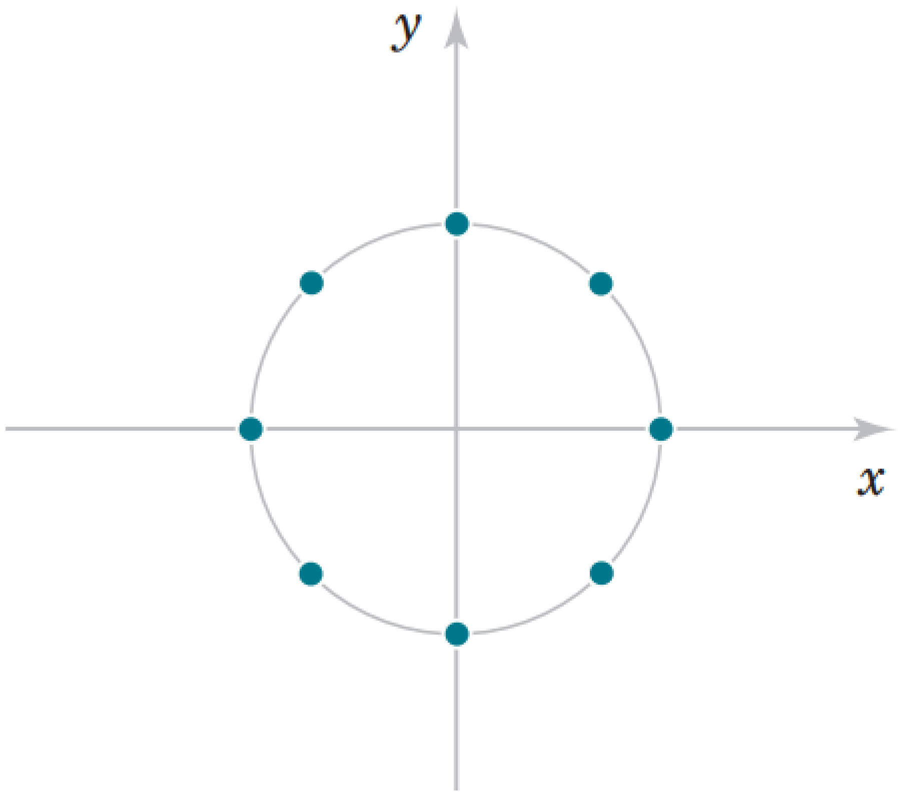


Linear Relationship (1/6)

Covariance (5/5):

<u>Zero covariance</u>





Linear Relationship (2/6)

Example1:

$f_{XY}(x,y)$

y \ x	**1**	**2**	**4**
3	$1/8$	0	0
4	$1/4$	0	0
5	0	$1/2$	0
6	0	0	$1/8$

Determine the covariance σ_{XY}?

$$cov(X,Y) = \sigma_{XY} = E(XY) - E(X)E(Y)$$

Linear Relationship (2/6)

Example1:

$f_{XY}(x, y)$

y \ x	**1**	**2**	**4**
3	1/8	0	0
4	1/4	0	0
5	0	1/2	0
6	0	0	1/8

Same as

x	1	1	2	4
y	3	4	5	6
$f_{XY}(x, y)$	**1/8**	**1/4**	**1/2**	**1/8**

Linear Relationship (3/6)

Example1 – Answer (1/6):

x	1	1	2	4
y	3	4	5	6
$f_{XY}(x,y)$	**1/8**	**1/4**	**1/2**	**1/8**

$f_{XY}(x,y)$

x	y	$f_{XY}(x,y)$	$x\,f_{XY}(x,y)$	$y\,f_{XY}(x,y)$	$xy\,f_{XY}(x,y)$
1	3	1/8			
1	4	1/4			
2	5	1/2			
4	6	1/8			

Linear Relationship (3/6)

Example1 – Answer (2/6):

x	y	$f_{XY}(x,y)$	$x\,f_{XY}(x,y)$	$y\,f_{XY}(x,y)$	$xy\,f_{XY}(x,y)$
1	3	1/8	1/8		
1	4	1/4	1/4		
2	5	1/2	2/2		
4	6	1/8	4/8		

Linear Relationship (3/6)

Example1 – Answer (3/6):

x	y	$f_{XY}(x,y)$	$x\,f_{XY}(x,y)$	$y\,f_{XY}(x,y)$	$xy\,f_{XY}(x,y)$
1	3	1/8	1/8	3/8	
1	4	1/4	1/4	4/4	
2	5	1/2	2/2	5/2	
4	6	1/8	4/8	6/8	

Linear Relationship (3/6)

Example1 – Answer (4/6):

x	y	$f_{XY}(x,y)$	$x\, f_{XY}(x,y)$	$y\, f_{XY}(x,y)$	$xy\, f_{XY}(x,y)$
1	3	1/8	1/8	3/8	3/8
1	4	1/4	1/4	4/4	4/4
2	5	1/2	2/2	5/2	10/2
4	6	1/8	4/8	6/8	24/8

Linear Relationship (3/6)

Example1 – Answer (5/6):

x	y	$f_{XY}(x,y)$	$x\, f_{XY}(x,y)$	$y\, f_{XY}(x,y)$	$xy\, f_{XY}(x,y)$
1	3	1/8	1/8	3/8	3/8
1	4	1/4	1/4	4/4	4/4
2	5	1/2	2/2	5/2	10/2
4	6	1/8	4/8	6/8	24/8
Sum			**15/8**	**37/8**	**75/8**
			$E(X)$	$E(Y)$	$E(XY)$

Example1 – Answer (6/6):

$\boldsymbol{E(X) = 15/8}$ $\boldsymbol{E(Y) = 37/8}$ $\boldsymbol{E(XY) = 75/8}$

$$cov(X,Y) = \sigma_{XY} = E(XY) - E(X)E(Y)$$

$$cov(X,Y) = \sigma_{XY} = \frac{75}{8} - \frac{(15)(37)}{64} = 0.703125$$

Positive covariance

Linear Relationship (4/6)

Correlation (1/2):

The correlation between the random variables X and Y is just scales the covariance by the product of the standard deviation of each variable. Correlation measures the strength of the relationship between variables and denoted as ρ_{XY}, where

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$-1 \le \rho_{XY} \le +1$$

Linear Relationship (4/6)

Correlation (2/2):

If X and Y are independent random variables,

$$\rho_{XY} = \sigma_{XY} = 0$$

However, if the correlation between two random variables is zero, we cannot conclude that the random variables are independent.

Linear Relationship (5/6)

Example2:

$f_{XY}(x, y)$

y \ x	1	2	4
3	1/8	0	0
4	1/4	0	0
5	0	1/2	0
6	0	0	1/8

Determine the correlation ρ_{XY}?

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Linear Relationship (6/6)

Example2 – Answer (1/4): (From the previous example)

x	y	$f_{XY}(x,y)$	$x\,f_{XY}(x,y)$	$y\,f_{XY}(x,y)$	$xy\,f_{XY}(x,y)$
1	3	1/8	1/8	3/8	3/8
1	4	1/4	1/4	4/4	4/4
2	5	1/2	2/2	5/2	10/2
4	6	1/8	4/8	6/8	24/8
Sum			**15/8**	**37/8**	**75/8**
			$E(X)$	$E(Y)$	$E(XY)$

Linear Relationship (6/6)

Example2 – Answer (2/4):

New

x	y	$f_{XY}(x,y)$	$x\,f_{XY}(x,y)$	$y\,f_{XY}(x,y)$	$xy\,f_{XY}(x,y)$	$x^2\,f_{XY}(x,y)$	$y^2\,f_{XY}(x,y)$
1	3	1/8	1/8	3/8	3/8	1/8	9/8
1	4	1/4	1/4	4/4	4/4	1/4	16/4
2	5	1/2	2/2	5/2	10/2	4/2	25/2
4	6	1/8	4/8	6/8	24/8	16/8	36/8
Sum			**15/8**	**37/8**	**75/8**	**35/8**	**177/8**
			$E(X)$	$E(Y)$	$E(XY)$	$E(X^2)$	$E(Y^2)$

Linear Relationship (6/6)

Example2 – Answer (3/4):

$E(X) = 15/8$ $E(Y) = 37/8$ $E(XY) = 75/8$

$E(X^2) = 35/8$ $E(Y^2) = 177/8$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$\sigma_X = \sqrt{V(X)} = \sqrt{E(X^2) - (E(X))^2} = \sqrt{0.859375}$$

$$\sigma_Y = \sqrt{V(Y)} = \sqrt{E(Y^2) - (E(Y))^2} = \sqrt{0.734375}$$

$$\sigma_{XY} = E(XY) - E(X)E(Y) = 0.703125$$

Linear Relationship (6/6)

Example2 – Answer (4/4):

$E(X) = 15/8$

$E(Y) = 37/8$

$E(XY) = 75/8$

$E(X^2) = 35/8$

$E(Y^2) = 177/8$

$$\rho_{XY} = \frac{0.703125}{\sqrt{(0.859375)(0.734375)}} = 0.885079$$

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #7: https://www.youtube.com/watch?v=APvrmntE_hA&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=9 Until the 32:51 minute

Thank You

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